Anthony Croft and Robert Davison

Foundation Maths

SEVENTH EDITION

Pearson

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Seventh edition

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Brief contents

Supporting resources

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Preface

Today, a huge variety of disciplines require their students to have knowledge of certain mathematical tools in order to appreciate the quantitative aspects of their subjects. At the same time, higher education institutions have widened access so that there is much greater variety in the pre-university mathematical experiences of the student body. Some students are returning to education after many years in the workplace or at home bringing up families.

Foundation Maths has been written for those students in higher education who have not specialised in mathematics at A or AS level. It is intended for non-specialists who need some but not a great deal of mathematics as they embark upon their courses of higher education. It is likely to be especially useful to those students embarking upon a Foundation Degree with mathematical content. It takes students from around the lower levels of GCSE to a standard which will enable them to participate fully in a degree or diploma course. It is ideally suited for those studying marketing, business studies, management, science, engineering, social science, geography, computer science combined studies and design. It will be useful for those who lack confidence and need careful, steady guidance in mathematical methods. Even for those whose mathematical expertise is already established, the book will be a helpful revision and reference guide. The style of the book also makes it suitable for those who wish to engage in self-study or distance learning.

We have tried throughout to adopt an informal, user-friendly approach and have described mathematical processes in everyday language. Mathematical ideas are usually developed by example rather than by formal proof. This reflects our experience that students learn better from examples than from abstract development. Where appropriate, the examples contain a great deal of detail so that the student is not left wondering how one stage of a calculation leads to the next. In *Foundation Maths*, objectives are clearly stated at the beginning of each chapter, and key points and formulae are highlighted throughout the book. Self-assessment questions are provided at the end of most sections. These test understanding of important features in the section and answers are given at the back of the book. These are followed by exercises; it is essential that these are attempted as the only way to develop competence and understanding is through practice. Solutions to these exercises are given at the back of the book and should be consulted only after the exercises have been attempted. We have included in many of the chapters a number of *challenge exercises*. These exercises are intentionally demanding and require a considerable depth of understanding. Solutions to these exercises can be found at **go.pearson .com/uk/he/resources**. A further set of test and assignment exercises is given at the end of each chapter. These are provided so that the tutor can set regular assignments or tests throughout the course. Solutions to these are not provided. Feedback from students who have used earlier editions of this book indicates that they have found the style and pace of the book helpful in their study of mathematics at university.

In order to keep the size of the book reasonable we have endeavoured to include topics which we think are most important, cause the most problems for students, and have the widest applicability. We have started the book with materials on arithmetic including whole numbers, fractions and decimals. This is followed by several chapters which gradually introduce important and commonly used topics in algebra. There follows chapters on sets, number bases and logic, collectively known as discrete mathematics. The remaining chapters introduce functions, trigonometry, vectors, matrices, complex numbers, statistics, probability and calculus. These will be found useful in the courses previously listed.

The best strategy for those using the book would be to read through each section, carefully studying all of the worked examples and solutions. Many of these solutions develop important results needed later in the book. It is then a good idea to cover up the solution and try to work the example again independently. It is only by doing the calculation that the necessary techniques will be mastered. At the end of each section the self-assessment questions should be attempted. If these cannot be answered then the previous few pages should be worked through again in order to find the answers in the text, before checking with answers given at the back of the book. Finally, the exercises should be attempted and, again, answers should be checked regularly with those given at the back of the book.

Foundation Maths is enhanced by video clips (see **go.pearson.com/uk/he/resources**) in which we, the authors, work through some algebraic examples and exercises taken from the book, pointing out techniques and key points. The icon VIDEO next to an exercise signifies that there is a corresponding video clip.

New to this 7th edition is the inclusion of many examples which illustrate how readily-available software can be used to tackle the mathematical problems you will meet in *Foundation Maths*. These examples are marked with symbol .

Although many mathematical software packages and apps are available, the ones used here for the purposes of illustration are Excel and GeoGebra. Further details of this important aspect are given on p. xxiii.

In conclusion, remember that learning mathematics takes time and effort. Carrying out a large number of exercises allows the student to experience a greater variety of problems, thus building up expertise and confidence. Armed with these the student will be able to tackle more unfamiliar and demanding problems that arise in other aspects of their course.

We hope that you find *Foundation Maths* useful and wish you the very best of luck.

Anthony Croft, Robert Davison 2020

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xxiv Advisory Committee on Mathematics Education: Mathematical Needs: Mathematics in the workplace and in Higher Education, June 2011. Advisory Committee on Mathematics Education.

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xxvi 9 13 87 88 109 295 297 298 341 344 353 354 358 377 464 476 502 GeoGebra: Screenshot of GeoGebra © GeoGebra 2019 **xxv 199 206 211 264 293 294 438 441 430 448 Microsoft Corporation:** Screenshot of Microsoft Excel © Microsoft 2019.

List of videos

The following table lists the videos which accompany selected exercises and examples in the book. You can view the videos at **go.pearson.com/uk/he/resources**

Name	Reference
Substitution of a value into a quadratic expression	Exercise 5.3 Q13
Simplification of expressions requiring use of the first law of indices	Exercise 6.1 Q8
Simplification of expressions requiring use of the second and third laws of indices	Exercise 6.1 Q10
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Removing the brackets from expressions 2	Example 7.24
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Solving simultaneous equations by elimination	Example 11.6
Solving a quadratic equation by factorisation	Example 11.10
Solving a quadratic equation using a formula	Example 11.16

Mathematical symbols

$+$	plus
$-$	minus
$\pm$	plus or minus
$\times$	multiply by
$\cdot$	multiply by
$\div$	divide by
$=$	is equal to
$\equiv$	is identically equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$<$	is less than
$\leqslant$	is less than or equal to
$\in$	is a member of set
ε	universal set
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\bar{A}$	complement of set A
$\subseteq$	subset
$\mathbb{R}$	all real numbers
$\mathbb{R}^+$	all numbers greater than 0
$\mathbb{R}^-$	all numbers less than 0
$\mathbb{Z}$	all integers
$\mathbb{N}$	all positive integers
$\mathbb{C}$	all complex numbers
$\mathbb{Q}$	rational numbers

$\prod$	irrational numbers
$\therefore$	therefore
∞	infinity
e	the base of natural logarithms (2.718 …)
ln	natural logarithm
log	logarithm of base 10
$\sum$	sum of terms
$\int$	integral
$\frac{\mathrm{d}y}{\mathrm{d}x}$	derivative of y with respect to x
π	‘pi’ $\approx$ 3.14159
$\neg$	negation (not)
$\wedge$	conjunction (and)
$\vee$	disjunction (or)
$\rightarrow$	implication

Using mathematical and statistical computer software and apps in *Foundation Maths*

Foundation Maths has been written for students taking further and higher education courses who have not specialised in mathematics on post-16 qualifications and who need to use mathematics or statistics in their courses. Our intentions are to provide a thorough, carefully-paced foundation in the mathematical methods needed for success, to develop understanding and to build confidence.

So what has computer software to do with *Foundation Maths*?

Computer software and apps available for use on tablets and smartphones which can be used to perform all of the mathematical and statistical calculations in *Foundation Maths* are now readily-available, either freely or at low-cost. You will probably come across a variety of such tools in your courses. They are able to go beyond arithmetical operations found on a calculator and can perform calculations using algebra. Two important topics that you will meet when studying calculus, namely differentiation and integration, give rise to problems which can be solved using software. Differentiation can be used to find the maximum and minimum values of a function, for example maximising profit or minimising cost in business analysis. Integration can be used, for example, in the solution of equations that describe the movement of a fluid or the vibration of an aerofoil. Software is used to apply these calculus techniques to such problems. Moreover, the software can produce visual representations of solutions which can supplement the information given in an algebraic answer and thereby provide more insight. For example, they can draw accurate graphs and enable the user to focus upon points of interest. Statistical apps can tackle the analysis of data sets and produce results in a wide variety of visually informative charts.

With all this software, why do I need to learn *Foundation Maths*?

At first sight it might appear that with access to these tools there is no longer a need to learn basic mathematical methods. However, and on the contrary, to be able to exploit their full capability and power, a firm understanding of the underlying mathematics is essential. In part, this is because to use such software requires the user to distinguish and understand mathematical and statistical terminology. For example, when using computer algebra software it is essential to understand the meaning of words such as *simplify*, *factorise*, *solve*, *expand*, …. Such words have precise mathematical meanings which inform important choices to be made by the user. Visualisation software is able to access user data in several ways (e.g. from a formula, from a set of data, from an external file) and display it using a variety of different graphs (for example, *polar*, *cartesian* and *logarithmic graphs*). So, it is important to understand what these words mean. Statistical software will allow you to interrogate large sets of data and to look for patterns in that data. You may be interested in whether two or more variables are associated, that is whether there is *correlation*, and if so, how strong is that association. Knowledge of the different ways in which this strength is measured, that is through *correlation coefficients*, is important if you are to make sensible choices and correct inferences when using the software. Whilst exceptionally powerful, software is not infallible! It is important that you can look critically at the output and make a judgement as to whether it is likely to be correct or not. Even when the output is correct, this output may be in a form that you do not recognise. Acquiring the mathematical fluency to compare and contrast different forms of output is a skill that you will develop by working through *Foundation Maths*.

To quote from the ACME[1] Mathematical Needs report: *It is sometimes argued that the advent of computers has reduced the need for people to be able to do mathematics. Nothing could be further from the truth. Off-the-shelf and purpose-designed computer software packages are creating ever more data sets, statistics and graphs. Working with mathematical models, which people need to be able to understand, interpret, interrogate and use advantageously, is becoming commonplace. The use of quantitative data is now omnipresent and informs workplace practice*.

So how might I want to use software as I work through this book?

Firstly, you are able to verify the solutions you have already obtained 'by hand', and this gives you confidence that your methods are appropriate and your solutions are correct. Secondly, you can explore the effect of changing some of the values or 'parameters' in a mathematical expression. For example, what will happen to the graphs of $y = x^2 + 2x + c$ or $y = (x - c)^2$ when c is varied from negative to zero and then to positive values. Investigations like these are straightforward when you have access to software, and the results can be illuminating and aid understanding. Thirdly, using software means that you can attempt more complicated, and often more realistic problems that would be too lengthy or time-consuming to tackle by hand. Finally, you are able to produce mathematical and statistical output, for example graphs or charts, in a way that looks attractive and professional.

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[1]ACME is the The Royal Society Advisory Committee on Mathematics Education, a distinguished body advising on mathematics education policy.

One of the purposes of this book is to help you to understand the mathematical foundations necessary to take advantage of this technology. We do not intend to teach you how to use the software – there are plenty of textbooks, user guides and on-line resources to help with this. However, we want to raise your awareness of what tools are available so that you become confident to explore these for yourself, or within your courses. Throughout the book we make reference to several pieces of software or apps outlined below. We do this solely for the purposes of illustration and are not making recommendations; there are numerous different tools available and we would encourage you to explore the field for yourself and to take advice from within your own institution.

The software and apps that we will refer to throughout *Foundation Maths*

We illustrate how *Microsoft Excel*[2] can be used for performing routine statistical calculations such as finding the mean and standard deviation of a set of data, for finding correlation coefficients and lines of best fit. Statistical charts and graphs such as the one shown in Figure i can be produced relatively easily from large sets of data.

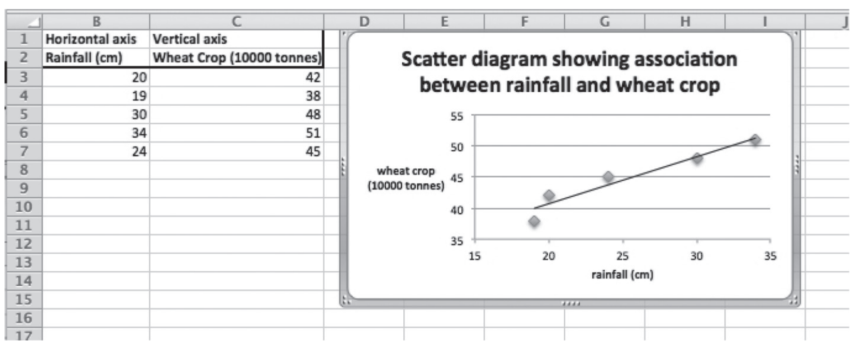

Figure i
Using *Microsoft Excel* for producing a statistical chart

We illustrate how *GeoGebra*[3] can be used to perform calculations arising in *algebra* and *calculus*, and how it can be used to explore important mathematical objects such as *vectors* and *matrices*. Figure ii shows a screenshot depicting several algebraic operations that you will learn about in *Foundation Maths*: commands for simplification, factorisation, expansion of brackets, and solving equations. Further examples are given as you work through the book.

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[2]Microsoft Excel is a spreadsheet developed by Microsoft Corporation

[3]GeoGebra is an interactive geometry, algebra, statistics and calculus application (**www.geogebra.org**)

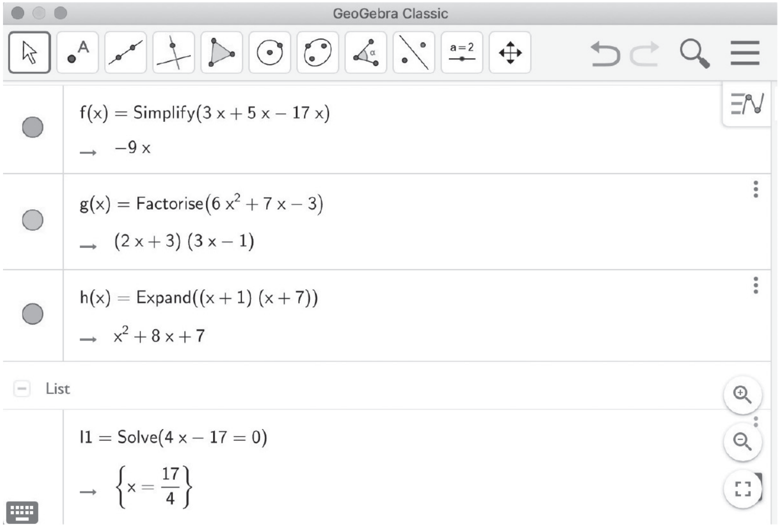

Figure ii
A selection of *GeoGebra* commands for algebraic manipulation

As we have noted earlier, there are many other commonly used software packages and apps. If you are studying, or intend to study, engineering, physics or mathematics you are likely to come across *Matlab*, *Mathematica* or *Maple*; these are extremely powerful technical computing systems that include additional toolboxes for tasks such as signal processing. If you are studying psychology or the social sciences it is likely that you will use statistical software such as *SPSS*, *minitab* or *R* in the analysis of large data sets. It will help your learning if you enquire about what packages are available for your use in the institution where you are studying and to explore how these can be put to use in the solution of exercises in *Foundation Maths*.

Examples illustrating use of software in *Foundation Maths*

The ability to use modern software in the solution of mathematical and statistical problems is an invaluable skill to develop. The first step in this development is to become aware of packages that are available, to appreciate how powerful they are, and how you can make use of them, particularly once you have acquired the necessary fundamental mathematical knowledge.

Throughout this edition of *Foundation Maths* we provide numerous illustrative examples of the wide-spread application and power of mathematical and statistical software. We would encourage you to try these and similar examples for yourself and to explore further.

Purpose	**Page**
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To explore the effect of changing A in $y = A \sin kx$, $y = A \cos kx$, $y = A \tan kx$	297
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Arithmetic of whole numbers

1

Objectives: This chapter:

- explains the rules for adding, subtracting, multiplying and dividing positive and negative numbers
- explains what is meant by an integer
- explains what is meant by a prime number
- explains what is meant by a factor
- explains how to prime factorise an integer
- explains the terms ‘highest common factor’ and ‘lowest common multiple’

1.1 Addition, subtraction, multiplication and division

Arithmetic is the study of numbers and their manipulation. A clear and firm understanding of the rules of arithmetic is essential for tackling everyday calculations. Arithmetic also serves as a springboard for tackling more abstract mathematics such as algebra and calculus.

The calculations in this chapter will involve mainly whole numbers, or **integers** as they are often called. The **positive integers** are the numbers

$$1, 2, 3, 4, 5 \dots$$

and the **negative integers** are the numbers

$$\dots \; -5, -4, -3, -2, -1$$

The dots (. . .) indicate that this sequence of numbers continues indefinitely. The number 0 is also an integer but is neither positive nor negative.

To find the **sum** of two or more numbers, the numbers are added together. To find the **difference** of two numbers, the second is subtracted from the first. The **product** of two numbers is found by multiplying the numbers together. Finally, the **quotient** of two numbers is found by dividing the first number by the second.

WORKED EXAMPLE

1.1

(a) Find the sum of 3, 6 and 4.

(b) Find the difference of 6 and 4.

(c) Find the product of 7 and 2.

(d) Find the quotient of 20 and 4.

Solution

(a) The sum of 3, 6 and 4 is

$$3 + 6 + 4 = 13$$

(b) The difference of 6 and 4 is

$$6 - 4 = 2$$

(c) The product of 7 and 2 is

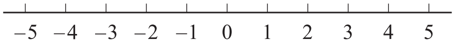
$$7 \times 2 = 14$$

(d) The quotient of 20 and 4 is $\frac{20}{4}$, that is 5.

When writing products we sometimes replace the sign $\times$ by '$\cdot$' or even omit it completely. For example, $3 \times 6 \times 9$ could be written as $3 \cdot 6 \cdot 9$ or (3)(6)(9).

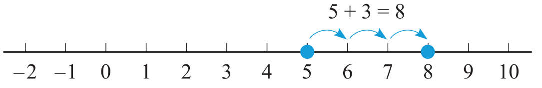
On occasions it is necessary to perform calculations involving negative numbers. To understand how these are added and subtracted consider Figure 1.1, which shows a **number line.**

Figure 1.1
The number line



Any number can be represented by a point on the line. Positive numbers are on the right-hand side of the line and negative numbers are on the left. From any given point on the line, we can add a positive number by moving that number of places to the right. For example, to find the sum $5 + 3$, start at the point 5 and move 3 places to the right, to arrive at 8. This is shown in Figure 1.2.

Figure 1.2
To add a positive number, move that number of places to the right



To subtract a positive number, we move that number of places to the left. For example, to find the difference $5 - 7$, start at the point 5 and move 7 places to the left to arrive at -2. Thus $5 - 7 = -2$. This is shown in Figure 1.3. The result of finding $-3 - 4$ is also shown to be -7.

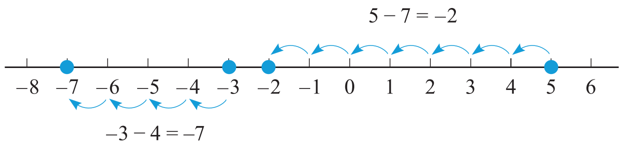


Figure 1.3 To subtract a positive number, move that number of places to the left

To add a negative number we move to the left. The result of finding $2 + (-3)$ is shown in Figure 1.4. Starting at 2, we move 3 places to the left, to arrive at -1.

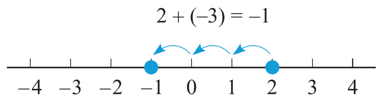


Figure 1.4 Adding a negative number involves moving to the left

We see that $2 + (-3) = -1$. Note that this is the same as the result of finding $2 - 3$, so that adding a negative number is equivalent to subtracting a positive number. For example

$$9+(-4) = 9-4 = 5, 3+(-7) = 3-7 = -4, -6+(-10) = -6-10 = -16$$

To subtract a negative number we move to the right. The result of finding $5 - (-3)$ is shown in Figure 1.5

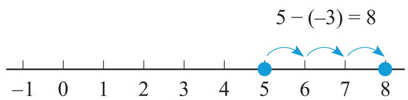


Figure 1.5 Subtracting a negative number involves moving to the right

We see that $5 - (-3) = 8$. This is the same as the result of finding $5+3$, so subtracting a negative number is equivalent to adding a positive number. For example

$$6-(-2) = 6+2 = 8, -5-(-3) = -5 + 3 = -2, -1-(-1) = -1+1 = 0$$

Key point

Adding a negative number is equivalent to subtracting a positive number.
Subtracting a negative number is equivalent to adding a positive number.

WORKED EXAMPLE

1.2 Evaluate (a) $8 + (-4)$, (b) $-15 + (-3)$, (c) $-15 - (-4)$.

Solution (a) $8 + (-4)$ is equivalent to $8 - 4$, that is 4.

(b) Because adding a negative number is equivalent to subtracting a positive number we find $-15 + (-3)$ is equivalent to $-15 - 3$, that is -18.

(c) $-15 - (-4)$ is equivalent to $-15 + 4$, that is -11.

When we need to multiply or divide negative numbers, care must be taken with the **sign** of the answer; that is, whether the result is positive or negative. The following rules apply for determining the sign of the answer when multiplying or dividing positive and negative numbers.

Key point

(positive) × (positive) = positive
(positive) × (negative) = negative
(negative) × (positive) = negative
(negative) × (negative) = positive

and

$$\frac{\text{positive}}{\text{positive}} = \text{positive}$$

$$\frac{\text{positive}}{\text{negative}} = \text{negative}$$

$$\frac{\text{negative}}{\text{positive}} = \text{negative}$$

$$\frac{\text{negative}}{\text{negative}} = \text{positive}$$

WORKED EXAMPLE

1.3

Evaluate

(a) $3 \times (-2)$ (b) $(-1) \times 7$ (c) $(-2) \times (-4)$ (d) $\frac{12}{(-4)}$ (e) $\frac{-8}{4}$ (f) $\frac{-6}{-2}$

Solution

(a) We have a positive number, 3, multiplied by a negative number, -2, and so the result will be negative:

$$3 \times (-2) = -6$$

(b) $(-1) \times 7 = -7$

(c) Here we have two negative numbers being multiplied and so the result will be positive:

$$(-2) \times (-4) = 8$$

(d) A positive number, 12, divided by a negative number, -4, gives a negative result:

$$\frac{12}{-4} = -3$$

(e) A negative number, −8, divided by a positive number, 4, gives a negative result:

$$\frac{-8}{4} = -2$$

(f) A negative number, −6, divided by a negative number, −2, gives a positive result:

$$\frac{-6}{-2} = 3$$

Self-assessment questions 1.1

1. Explain what is meant by an integer, a positive integer and a negative integer.
2. Explain the terms sum, difference, product and quotient.
3. State the sign of the result obtained after performing the following calculations:
   (a) $(-5) \times (-3)$ (b) $(-4) \times 2$ (c) $\frac{7}{-2}$ (d) $\frac{-8}{-4}$.

Exercise 1.1

1. Without using a calculator, evaluate each of the following:
   (a) $6 + (-3)$ (b) $6 - (-3)$
   (c) $16 + (-5)$ (d) $16 - (-5)$
   (e) $27 - (-3)$ (f) $27 - (-29)$
   (g) $-16 + 3$ (h) $-16 + (-3)$
   (i) $-16 - 3$ (j) $-16 - (-3)$
   (k) $-23 + 52$ (l) $-23 + (-52)$
   (m) $-23 - 52$ (n) $-23 - (-52)$
2. Without using a calculator, evaluate
   (a) $3 \times (-8)$ (b) $(-4) \times 8$ (c) $15 \times (-2)$
   (d) $(-2) \times (-8)$ (e) $14 \times (-3)$
3. Without using a calculator, evaluate
   (a) $\frac{15}{-3}$ (b) $\frac{21}{7}$ (c) $\frac{-21}{7}$ (d) $\frac{-21}{-7}$ (e) $\frac{21}{-7}$
   (f) $\frac{-12}{2}$ (g) $\frac{-12}{-2}$ (h) $\frac{12}{-2}$
4. Find the sum and product of (a) 3 and 6, (b) 10 and 7, (c) 2, 3 and 6.
5. Find the difference and quotient of (a) 18 and 9, (b) 20 and 5, (c) 100 and 20.

1.2 The BODMAS rule

When evaluating numerical expressions we need to know the order in which addition, subtraction, multiplication and division are carried out. As a simple example, consider evaluating $2 + 3 \times 4$. If the addition is carried out first we get $2 + 3 \times 4 = 5 \times 4 = 20$. If the multiplication is carried out first we get $2 + 3 \times 4 = 2 + 12 = 14$. Clearly the order of carrying out numerical operations is important. The BODMAS rule tells us the order in which we must carry out the operations of addition, subtraction, multiplication and division.

Key point

BODMAS stands for

Brackets ( )	First priority
Of $\times$	Second priority
Division $\div$	Second priority
Multiplication $\times$	Second priority
Addition $+$	Third priority
Subtraction $-$	Third priority

This is the order of carrying out arithmetical operations, with bracketed expressions having highest priority and subtraction and addition having the lowest priority. Note that 'Of', 'Division' and 'Multiplication' have equal priority, as do 'Addition' and 'Subtraction'. 'Of' is used to show multiplication when dealing with fractions: for example, find $\frac{1}{2}$ of 6 means $\frac{1}{2} \times 6$.

If an expression contains only multiplication and division, we evaluate by working from left to right. Similarly, if an expression contains only addition and subtraction, we also evaluate by working from left to right.

WORKED EXAMPLES

1.4 Evaluate

(a) $2 + 3 \times 4$ (b) $(2 + 3) \times 4$

Solution

(a) Using the BODMAS rule we see that multiplication is carried out first. So

$$2 + 3 \times 4 = 2 + 12 = 14$$

(b) Using the BODMAS rule we see that the bracketed expression takes priority over all else. Hence

$$(2 + 3) \times 4 = 5 \times 4 = 20$$

1.5 Evaluate

(a) $4 - 2 \div 2$ (b) $1 - 3 + 2 \times 2$

Solution (a) Division is carried out before subtraction, and so

$$4 - 2 \div 2 = 4 - \frac{2}{2} = 3$$

(b) Multiplication is carried out before subtraction or addition:

$$1 - 3 + 2 \times 2 = 1 - 3 + 4 = 2$$

1.6 Evaluate

(a) $(12 \div 4) \times 3$ (b) $12 \div (4 \times 3)$

Solution Recall that bracketed expressions are evaluated first.

(a) $(12 \div 4) \times 3 = \left(\frac{12}{4}\right) \times 3 = 3 \times 3 = 9$

(b) $12 \div (4 \times 3) = 12 \div 12 = 1$

Example 1.6 shows the importance of the position of brackets in an expression.

Self-assessment questions 1.2

1. State the BODMAS rule used to evaluate expressions.
2. The position of brackets in an expression is unimportant. True or false?

Exercise 1.2

1. Evaluate the following expressions:
   (a) $6 - 2 \times 2$ (b) $(6 - 2) \times 2$
   (c) $6 \div 2 - 2$ (d) $(6 \div 2) - 2$
   (e) $6 - 2 + 3 \times 2$ (f) $6 - (2 + 3) \times 2$
   (g) $(6 - 2) + 3 \times 2$ (h) $\frac{16}{-2}$ (i) $\frac{-24}{-3}$
   (j) $(-6) \times (-2)$ (k) $(-2)(-3)(-4)$

2. Place brackets in the following expressions to make them correct:
   (a) $6 \times 12 - 3 + 1 = 55$
   (b) $6 \times 12 - 3 + 1 = 68$
   (c) $6 \times 12 - 3 + 1 = 60$
   (d) $5 \times 4 - 3 + 2 = 7$
   (e) $5 \times 4 - 3 + 2 = 15$
   (f) $5 \times 4 - 3 + 2 = -5$

1.3 Prime numbers and factorisation

A **prime number** is a positive integer, larger than 1, which cannot be expressed as the product of two smaller positive integers. To put it another way, a prime number is one that can be divided exactly only by 1 and itself.

For example, $6 = 2 \times 3$, so 6 can be expressed as a product of smaller numbers and hence 6 is not a prime number. However, 7 is prime. Examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23. Note that 2 is the only even prime.

Factorise means 'write as a product'. By writing 12 as 3×4 we have factorised 12. We say 3 is a **factor** of 12 and 4 is also a factor of 12. The way in which a number is factorised is not unique: for example, 12 may be expressed as 3×4 or 2×6. Note that 2 and 6 are also factors of 12.

When a number is written as a product of prime numbers we say the number has been **prime factorised.**

To prime factorise a number, consider the technique used in the following examples.

WORKED EXAMPLES

1.7 Prime factorise the following numbers:

(a) 12 (b) 42 (c) 40 (d) 70

Solution

(a) We begin with 2 and see whether this is a factor of 12. Clearly it is, so we write

$$12 = 2 \times 6$$

Now we consider 6. Again 2 is a factor so we write

$$12 = 2 \times 2 \times 3$$

All the factors are now prime, that is the prime factorisation of 12 is $2 \times 2 \times 3$.

(b) We begin with 2 and see whether this is a factor of 42. Clearly it is and so we can write

$$42 = 2 \times 21$$

Now we consider 21. Now 2 is not a factor of 21, so we examine the next prime, 3. Clearly 3 is a factor of 21 and so we can write

$$42 = 2 \times 3 \times 7$$

All the factors are now prime, and so the prime factorisation of 42 is $2 \times 3 \times 7$.

(c) Clearly 2 is a factor of 40,

$$40 = 2 \times 20$$

Clearly 2 is a factor of 20,

$$40 = 2 \times 2 \times 10$$

Again 2 is a factor of 10,

$$40 = 2 \times 2 \times 2 \times 5$$

All the factors are now prime. The prime factorisation of 40 is $2 \times 2 \times 2 \times 5$.

(d) Clearly 2 is a factor of 70,

$$70 = 2 \times 35$$

We consider 35: 2 is not a factor, 3 is not a factor, but 5 is:

$$70 = 2 \times 5 \times 7$$

All the factors are prime. The prime factorisation of 70 is $2 \times 5 \times 7$.

1.8 Prime factorise 2299.

Solution We note that 2 is not a factor and so we try 3. Again 3 is not a factor and so we try 5. This process continues until we find the first prime factor. It is 11:

$$2299 = 11 \times 209$$

We now consider 209. The first prime factor is 11:

$$2299 = 11 \times 11 \times 19$$

All the factors are prime. The prime factorisation of 2299 is $11 \times 11 \times 19$.

Here we illustrate how software can be used to perform the prime factorisation of 2299 using GeoGebra Classic v6 with the CAS (Computer Algebra System) interface. Numerous other packages are available that perform the same calculation. Figure 1.6 illustrates the output following the command **`Factor( )`** which performs the prime factorisation. Compare this output to the solution of Worked Example 1.8, particularly the use of the power 2 in 11^2. Powers, or indices as they are also called, are explained in Chapter 5.

Figure 1.6
Syntax used to perform prime number calculations.

In addition, Figure 1.6 shows the commands **`IsPrime( )`** and **`NextPrime( )`** which will test whether a given number is prime and find the first prime number greater than a given number. You should consult the on-line help provided with your software to explore other prime number commands.